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TECHNICAL REPORT BRL-TR-3192

BRL

AD-A231 621

VARIATIONAL METHOD IN THE STATISTICAL THEORY OF TURBULENCE

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JANUARY 1991

U.S. ARMY LABORATORY COMMAND

BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

NOTICES

UNCLASSIFIED

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
January 1991

3. REPORT TYPE AND DATES COVERED
Final Jan 88 - Jun 90

4. TITLE AND SUBTITLE
Variational Method in the Statistical Theory of Turbulence

5. FUNDING NUMBERS
STAS # DAAL03-86-D-0001
1L161102AH43

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
USA Ballistic Research Laboratory
ATTN: SLCBR-DD-T
Aberdeen Proving Ground, MD 21005-5066

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
BRL-TR-3192

11. SUPPLEMENTARY NOTES
*Names appear alphabetically.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for Public Resease - Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A new approach to the calculation of turbulent flows is presented. Variational principles are introduced into the statistical theory of turbulence enabling the determination of correlation functions by extremizing certain functionals. To demonstrate the efficiency of the technique a sample calculation of a mean flow profile and certain two point correlation functions of a cylindrically symmetric free jet were carried out using the Rayleigh-Ritz method. A judicious choice of the trial functions leads to the determination of the correlation functions with a modest amount of computation. The results are in reasonable agreement with experimental data.

14. SUBJECT TERMS
Two Phase Flow, Turbulence, Statistical Turbulence

15. NUMBER OF PAGES
33

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

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TABLE OF CONTENTS

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1. INTRODUCTION

The current generation of interior ballistic codes rely on turbulence models which incorporate little if any of the underlying physics of the gas–particle motion in a gun tube. Indeed, more often than not, the fictitious length scales and adjustable constants employed are based on experiments which have no relationship to the flow being modeled. The result at best is post predictive in nature and no claim to physical representation can be made. To put turbulence sub–models in a ballistic setting on a more realistic footing, this study was undertaken. A new mathematical approach to the problem is presented and results for the single phase, turbulent case are given. In a companion paper the flow in a two–phase jet is analyzed and compared to data obtained at BRL.

Turbulence is prevalent in practically all naturally occurring flow processes and has challenged, without adequate resolution, the best scientific minds for over a century. The problem is especially hard due to the very large number of coupled degrees of freedom involved. Hence the description of such fluid flows cannot proceed *via* usual techniques of approximating the system by a linear one and using perturbation techniques around the linearized limit. Despite the existence of an enormous body of theoretical literature on the subject, there is still no "standard" theoretical treatment of fully developed turbulence. In our previous works we took a statistical approach to this problem (Domokos, Domokos–Kovesi and Zoltani 1988), following the pioneering works of Martin, Siggia, Rose (1973) and De Dominicis and Peliti (1978). In this approach one considers the classical equations of motion of a fluid, *viz.* the Navier–Stokes equations, perturbed by a random force. From the physical point of view, the random force represents the fluctuations in the fluid; its presence is necessary in order to avoid unstable, non–turbulent solutions of the equations of motion. The statistics of the random force, in turn, generates a statistical distribution of the components of the flow velocity. The correlation functions computed from the latter distribution can be compared directly with the results of measurements. An application of the technique developed by these authors to the case of turbulent channel flow led to a reasonable agreement with the available experimental data, cf. Burgett (1989). The approach adopted there consisted of a combination of analytical and numerical methods: an analytic approach was followed and numerical computations were used only for the computation of Fourier transforms, integrations, etc. While this approach led to a considerable reduction of computational complexity, it became clear that a straightforward application of those techniques to problems like fluid flow in jets would lead to an unreasonably large amount of computing time.

There are basically two features of the approach which make the computations difficult. First, there is no easy way available to satisfy the constraint imposed by the equation of continuity on the various velocity correlation functions; second, the approximation methods used are essentially perturbative in nature, hence, one has to labor very hard in order to achieve a reasonable accuracy.

In this work we describe some further developments in the statistical theory of fully developed turbulence which are designed to avoid the difficulties outlined above. First, we describe a technique by means of which the equation of continuity can be automatically satisfied. (We explicitly describe the procedure for the case when the fluid may be regarded an incompressible one, although the method can be generalized to arbitrary, compressible flows.) Second, we develop a variational approach to the computation of correlation functions. This enables us to avoid perturbative approaches altogether.

The paper is organized as follows. In the next section we briefly review the statistical approach to turbulent flows as described in Domokos, Domokos–Kovesi and Zoltani (1988). In Sec. 3 we introduce vector potentials which enable us to satisfy the equation of continuity identically. We also discuss the method of imposing various symmetry requirements on the correlation functions. We work out explicitly the constraints imposed by the requirement of cylindrical symmetry. The variational approach is described in Sec. 3.2. A sample calculation is presented in Sec. 4: some correlation functions of an axisymmetric jet are calculated by means of the techniqe developed here. Sec. 5 contains a discussion of the results.

2. REVIEW OF THE STATISTICAL APPROACH

Throughout this paper, a condensed notation is used. We consider a class of systems described by a classical equation of motion of the general form:

$$\partial_t X + F[X] = f(x). \tag{2.1}$$

Here X stands for an element of the vector space of dynamical variables. F[X] is an autonomous map of the vector space upon itself; x stands for both spatial coordinates and the time variable, $x = (\mathbf{x},t)$. Finally, f(x) represents a Gaussian random force acting upon

the system; it plays the role of a disordering field, driving the system described by eq. (2.1) away from a non–chaotic behavior. Specifically, if the system is an incompressible fluid described by the Navier–Stokes equations, X is identified with the velocity field, **u**. The map F[X] reads:

$$F[u]^i = (u^s \partial_s) u^i + \partial^i p - \nu \nabla^2 u^i , \tag{2.2}$$

with the velocity field satisfying the constraint, $\nabla . \mathbf{u} = 0$. In that case, the quantity p (the pressure divided by the density), is not an independent dynamical variable: it can be explicitly expressed in terms of **u**. Due to the constraint upon **u**, we are free to assume that the perturbing random force is solenoidal.

The quantity playing the central role in theories of this type is the generating functional of the correlation functions. It can be expressed in terms of the probability distribution of the random force as follows. Let K stand for the correlation operator of the random force. Then the generating functional of the correlation functions of X is given by the functional integral:

$$Z[j] = \int DX \exp - [((\partial_t X - F), K(\partial_t X - F)) + (j,X)], \tag{2.3}$$

where (.,.) stands for a scalar product over the vector space, involving integration over space–time variables and summation over tensor indices, whereas j is an arbitrary function: the functional derivatives of Z with respect to j give the correlation functions. The cumulants are generated by the functional W = –lnZ. Letting f to be a white noise, *viz.* $K(x_1,x_2) = k\ \delta(t_1 - t_2)\delta^3(\mathbf{x}_1 - \mathbf{x}_2)$ (where k is a constant) is often a satisfactory choice.

The functional weight of integration, DX, is proportional to the (infinite) determinant,

$$\mathrm{Det}\ (\partial_t - \delta F / \delta X). \tag{2.4}$$

It was shown by Domokos, Domokos–Kovesy and Zoltani (1988) that the latter can be expressed in terms of a functional integral over Fadeev–Popov ghosts (Itzykson and Zuber

1980). The central questions are: what is a satisfactory implementation of the constraints imposed upon X (in the case of an incompressible fluid flow, the constraint $\nabla . \mathbf{u} = 0$) as well as the development of suitable approximation techniques for the computation of Z (or its functional derivatives).

3. THE MATHEMATICAL APPROACH

3.1 Incompressible Flow: Vector Potentials and Symmetries. An incompressible flow is characterized by the fact that the velocity field is solenoidal. Any solenoidal vector field is obtained as the curl of a vector potential. Hence, we write, $\mathbf{u} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. It is also well known that, given the velocity field, the vector potential is determined only up to the gradient of an arbitrary scalar function: $\mathbf{A}$ and $\mathbf{A} + \nabla$ H give rise to the the same velocity field. (This is called the gauge freedom in electromagnetic theory: we adopt the same terminology here.) The scalar function, H can be chosen so as to simplify the problem of determining the vector potential. The equation satisfied by the vector potential is obtained by substituting the relationship, $\mathbf{u} = \nabla \times \mathbf{A}$ into the Navier–Stokes equation. This is straight forward and we do not reproduce the result here. Correspondingly, in the statistical theory discussed by Domokos, Domokos–Kovesi and Zoltani (1988) and briefly reviewed in the preceding section, one first determines the correlation functions of the vector potential; the velocity correlation functions are then obtained by taking the curl with respect to every argument.

It is worth remarking that this way of satisfying the incompressibility constraint is not the only possibility: it would be possible to take the constraint into account by means of the Fadeev–Popov method, see Itzykson and Zuber (1980). However, the method described here is simpler and leads to results more quickly than any other approach we know of. This is due to the gauge freedom just described.

In what follows, we use a gauge such that the component of the vector potential along the mean flow vanishes. ("Axial gauge".) On denoting the mean flow by U, the condition to be satisfied is: $\mathbf{U}.\mathbf{A} = 0$. This can always be achieved. In fact, let B be a vector potential which reproduces the velocity field , but its projection onto U is not necessarily zero. Then, in order to satisfy $\mathbf{U}.\mathbf{A} = 0$, with $\mathbf{A} = \mathbf{B} + \nabla \mathbf{H}$, the scalar H has to satisfy the differential equation,

$$\mathbf{U}.\mathbf{B} + \mathbf{U}.\nabla \mathbf{H} = 0. \tag{3.1.1}$$

There remains a residual gauge freedom, *viz.* a function h satisfying $\mathbf{U}.\nabla h = 0$ can always be added to any solution of (3.1).

Any symmetry requirement should be now imposed upon the correlation functions of the vector potentials instead of the correlation functions of the velocity field itself. A symmetry of a flow means that the correlation functions are *invariant* under a subgroup of the Euclidean group. For instance, homogeneous turbulence means invariance under translations, hence, the n–point correlation function of the vector potential depends on the n–1 coordinate differences only, not on the coordinates themselves. Likewise, in the case of isotropic turbulence, the correlation functions of order 2n are proportional to the n–fold direct product of the metric tensor with itself, whereas correlation functions of odd order vanish.

Let us now concentrate on the two point correlation function; the construction of the general n–point correlation function proceeds in a similar fashion. For the sake of definiteness, we choose a coordinate system such that its third axis coincides with the direction of the mean flow, U. Due to the fact that the relationship between the vector potential and the velocity field is a linear one, a Reynolds decomposition of the vector potential leads to one of the velocity field. Let us write,

$$\begin{aligned} \mathbf{A} &= \langle \mathbf{A} \rangle + \mathbf{a}, \\ \mathbf{U} &= \nabla \times \langle \mathbf{A} \rangle, \\ \langle \mathbf{a} \rangle &= 0, \\ \mathbf{u}' &= \nabla \times \mathbf{a}, \end{aligned} \tag{3.1.2}$$

where $\mathbf{u}'$ stands for the fluctuating part of the velocity field and $\langle ... \rangle$ denotes, as usual, the expectation value of a quantity. In an axial gauge we have:

$$A_3 = a_3 = 0, \tag{3.1.3}$$

so that only the components of the vector potential transverse to the mean flow are nonvanishing. (These components are denoted by subscripts/superscripts in capital letters, A, B, etc.)

The two–point correlation function of the vector potential $\mathbf{a}(\mathbf{x},t)$ can be decomposed in a basis of second rank tensors in the plane perpendicular to the mean flow. A basis of

such tensors consists of three elements. Correspondingly, the two–point correlation function can be written as follows:

$$Z_{AB} \equiv \langle a_A(\mathbf{x}_1,t_1)\, a_B(\mathbf{x}_2,t_2)\rangle = \delta_{AB}\, Z_1 + \epsilon_{AB}\, \epsilon_{RS} x_1^{\ R} x_2^{\ S}\, Z_2 + \tfrac{1}{2}(x_{1_A} x_{2_B} + x_{2_A} x_{1_B})\, Z_3, \tag{3.1.4}$$

where δ_{AB} and ϵ_{AB} are the Kronecker and Levi–Civita tensors, respectively. The functions Z_1, Z_2 and Z_3 are invariant under rotations around the third axis and they are symmetric functions of their arguments, $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2)$. (The antisymmetric product of the two coordinates has been inserted in front of Z_2 so as to satisfy the permutation symmetry of the correlation function with all three invariant functions being symmetric.)

Thus we found that the two point correlation tensor of an incompressible fluid has three independent components only. Cylindrical symmetry of the flow further reduces the number of independent elements to two. In fact, cylindrical symmetry means that the correlation function is invariant under rotations around the third axis; the tensors ϵ and δ are the only invariant ones under such rotations; hence $Z_3 = 0$.

We now give the explicit expressions of the velocity correlation functions for a flow of cylindrical symmetry.

$$\begin{aligned} G_{AB} &= \delta_{AB}\, \partial_{13}\partial_{23}\, Z_1, \\ G_{A3} &= -\, \partial_{13}[x_{1_A} Z_2 + \partial_{2_A} Z_1], \\ G_{33} &= \delta^{RS} \partial_{1_R} \partial_{2_S} Z_1 + [2 + x_1^{\ R} \partial_{1_R} + x_2^{\ R} \partial_{2_R}] Z_2. \end{aligned} \tag{3.1.5}$$

The notation used in this and subsequent equations is the following. The first subscript denotes the argument in the correlation function; the second subscript or superscript refers to the vector component. The summation convention is used throughout.

Cylindrically symmetric turbulent flows of incompressible fluids have been, of course, discussed previously, cf. Batchelor (1946), Chandrasekhar (1950), Trevino (1982). Where

they overlap, our results agree with those of these works. The present approach is, however, more general, since it can be applied to flows of arbitrary symmetry and it is more explicit than the treatment of Trevino (1982).

Finally, we just mention that the present construction can be generalized in a straight forward fashion to compressible flows where both the velocity field and the density field are dynamical variables. In that case one has to introduce two vector potentials with a correspondingly enlarged group of gauge transformations; this problem will be discussed elsewhere.

3.2 Variational Principles for the Correlation Functions. In this Section we return to the condensed notation used in Sec. 2. Consider the expression of the generating functional of the cumulants, as quoted there. We have:

$$W[j] = -\ln \int DX \exp - [((\partial_t - F),K(\partial_t - F)) + (j,X)], \tag{3.2.1}$$

The averages of the various quantities are given by functional derivatives of W,

$$G(1) \equiv \langle X(x_1)\rangle = \frac{\delta W}{\delta j(x_1)},$$

$$G(1,2) \equiv \langle X(x_1)\, X(x_2)\rangle = -\frac{\delta^2 W}{\delta j(x_1)\, \delta j(x_2)} + G(1)G(2)\,, \tag{3.2.2}$$

etc. Instead of the functional argument j, we can now introduce G(1) as a functional argument, by means of a Legendre transformation in function space. Let us define a new functional,

$$S_1 = W - (j,G_1). \tag{3.2.3}$$

One readily verifies with the help of (3.2.1) that S_1 is a functional of G(1), its first functional derivative being given by the expression:

$$\frac{\delta S_1}{\delta G(1)} = -j(1) \tag{3.2.4}$$

The functional S_1 can be determined from a functional differential equation obtained by combining eqs. (3.2.1) thru (3.2.4). This gives:

$$S_1 - (G_1, \frac{\delta S_1}{\delta j})$$

$$= -\ln \int DX \exp - [((\partial_t X - F), K(\partial_t X - F)) - (\frac{\delta S_1}{\delta j}, X)]. \tag{3.2.5}$$

The differential equation can be solved by iteration; one usually takes as a zeroth approximation the functional:

$$S_1^0 = S[G_1] \equiv ((\partial_t G_1 - F[G_1]), K(\partial_t G_1 - F[G_1]) \tag{3.2.6}$$

The principal advantage of the functional S_1 is that it becomes stationary if the external "source", j, is put equal to zero, see eq. (3.2.4). hence, one may determine S_1 up to a certain accuracy, but thereafter a calculation of G_1 does not have to rely upon any perturbative approximation scheme. This procedure can be generalized if one is interested in determining the higher order correlation functions as well as the average of the dynamical variable itself. We outline the procedure for a scheme aimed at determining the functions G_1 and G_2. It is necessary to introduce a bilinear source, h(1,2), in addition to j, *viz.*

$$W[j,h] = -\ln \int DX \exp - [((\partial_t X - F), K(\partial_t X - F)) + (j,X) + (X,hX)] \tag{3.2.7}$$

Similarly to the procedure outlined above, one now performs a double Legendre transform in order to obtain a functional, $S_2[G_1,G_2]$:

$$S_2 = W[j,h] - (G_1,j) - \mathrm{Tr}(hG_2), \tag{3.2.8}$$

where the trace is understood in the operator sense. One readily verifies the relations,

$$\frac{\delta S_2}{\delta G_2} = - h,$$
$$\frac{\delta S_2}{\delta G_1} = - j. \tag{3.2.9}$$

Thus, S_2 is stationary in both G_1 and G_2 in the limit of vanishing sources. The equation obeyed by the functional S_2 can be determined in the same way as eq. (3.2.5). We merely quote the result:

$$S_2 - (G_1,\frac{\delta S_2}{\delta G_1}) - \mathrm{Tr}(G_2\frac{\delta S_2}{\delta G_2})$$
$$= - \ln \int DX \exp - [((\partial_t X - F),K(\partial_t X - F)) - (\frac{\delta S_2}{\delta G_1},X) - (X,\frac{\delta S_2}{\delta G_2}X)]. \tag{3.2.10}$$

Just as for the equation obeyed by S_1, the only known method of solving (3.2.10) is by iteration. We give here the first approximation to S_2:

$$S_2{}^1 = S[G_1] + \mathrm{Tr}(\frac{\delta^2 S}{\delta G_1 \delta G_1}G_2) - \mathrm{Tr}(\ln G_2). \tag{3.2.11}$$

(This form of $S_2{}^1$ is obtained by changing to an integration variable, $Y = X - G_1$ in eq. (3.2.10); thereafter $S[Y + G_1]$ is expanded in powers of Y up to quadratic terms. The functional differential equation is then solved by quadrature.)

Functionals of the type (3.2.10) were first used by De Dominicis and Martin (1964) and Domokos and Suranyi (1964); the differential equations obeyed by such functionals was obtained by Cornwall, Jackiw and Tomboulis (1974). In the papers just quoted, it is also shown that higher order correlation functions can be computed by the application of either one of the variational principles described above. In this work, however, we concentrate upon the average flow (corresponding to G_1) and the two point correlation function. For this purpose, the use of the variational principle based on the functional S_2 is best suited.

4. CALCULATION OF A CYLINDRICALLY SYMMETRIC JET

As an illustration of the technique developed above, we now perform a computation of some properties of a free jet, with a cylindrically symmetric mean flow. We use the variational principle (3.2.8) and compute the functional S_2 in the first approximation, cf. eq. (3.2.11). The power of the variational principle lies in the fact that one can use a Rayleigh–Ritz method in order to minimize S_2. This leads to a considerable reduction in the complexity of computation: instead of solving coupled partial differential equations for the correlation functions, one tries to guess the form of the vector potential for the mean flow and of the functions Z_1 and Z_2 in (3.1.5), leaving a few free parameters. On substituting this form into the expression of S_2, the problem reduces to the computation of integrals and the minimization of the expression so obtained as a function of the parameters. The success of such an approach depends on finding a reasonable funtional form of $<A>$ and of Z_1 and Z_2. This can be accomplished only by a judicious use of one's physical intuition and by trial and error.

In this calculation we want to simplify the calculation as much as possible: we guess some simple functional forms which could reproduce the main features of the experimental data.

Let us examine now the Navier–Stokes operator entering the expression of the functional S_2:

$$N^i[u] = \partial_t u^i - \nu\nabla^2 u^i + (u^s \partial_s) u^i + \partial^i p \tag{4.1}$$

We are interested in fully developed turbulence: that means that it is described by a stationary ensemble. Hence the term containing the time derivative can be omitted from (4.1). Next we notice that if one introduces dimensionless quantities (as we shall do in what follows), the coefficient of the viscous term, $\nabla^2 \mathbf{u}$, becomes proportional to $1/\mathrm{Re}$, where Re stands for the Reynolds number. Hence, for flows of large Reynolds numbers ($\mathrm{Re} \approx 10^4$, say), it is safe to omit the viscous term too, as long as we are interested in the behavior of the fluid on scales substantially larger than the dissipation scale. Thus, we are going to work with the truncated Navier–Stokes operator,

$$n^i[\mathbf{u}] + \partial^i p, \tag{4.2}$$

with

$$n^i[\mathbf{u}] = u^s \partial_s u^i.$$

The next task is to eliminate the pressure from the generating functional. To this end, we regard the pressure as one of the fluctuating dynamical variables and carry out the functional integration over it explicitly. Using an operator notation as before, we have to compute a functional integral of the form:

$$\int Dp \exp-\tfrac{1}{2}[\mathbf{n} + \nabla p]\, K\, [\mathbf{n} + \nabla p]. \tag{4.3}$$

This is a standard Gaussian integral. The result of the computation, to be inserted into the expression of the functional S_2, is the following:

$$\int Dp \exp-\tfrac{1}{2}[\mathbf{n} + \nabla p]\, K\, [\mathbf{n} + \nabla p] = \exp-\tfrac{1}{2}\mathbf{n}\, K^T\, \mathbf{n}, \tag{4.4}$$

with

$$K^T = K - (\nabla K)\tilde{}\ [\nabla K \nabla + \alpha^2]^{-1}\, \nabla K, \tag{4.5}$$

where $\tilde{}$ denotes the transpose of an operator.

In the last equation we inserted a constant α^2 in order to make the inverse of the operator $\nabla K \nabla$ well defined on large length scales (equivalently, at small wave numbers). Evidently, K^T is the transversepart of K, $\nabla K^T = 0$. We now have to make a physical assumption about the correlation operator, K. We argue that the stirring force should point in the direction of the mean flow: this is both intuitively plausible and it is the simplest way of satisfying the requirement of cylindrical symmetry of the problem. Otherwise, we assume the the correlation to be of short range. Hence we take, (cf. Sec. 2):

$$K_{ij}(\mathbf{x},\mathbf{x}') = k\, \delta_{i3}\, \delta_{j3}\, \delta^3(\mathbf{x}-\mathbf{x}'). \tag{4.6}$$

In order to obtain a finite result even in the limit $\alpha \to 0$, we have to assume that k is proportional to $1/\alpha$. With this, the transverse part of K becomes:

$$K^T_{ij}(\mathbf{x},\mathbf{x}') = a\, e^{-\alpha|z - z'|}\, \delta^2(\mathbf{x}^A - \mathbf{x}'^A)\delta_{i3}\delta_{j3}, \tag{4.7}$$

where a is a constant independent of α. At the end of the calculation one may take $\alpha \to 0$, although there is some evidence that a finite α (just as in an Ornstein–Zernike process) gives slightly better results, cf. Burgett (1989), Ch.6.

We are now ready to compute the functional $S_2{}^1$, eq. (3.2.11). Clearly, the first term reads:

$$S[U] = \int d^3x d^3x'\, n^i[U(\mathbf{x})]K^T_{ij}(\mathbf{x},\mathbf{x}')n^j[U(\mathbf{x}')] \tag{4.8}$$

On expressing the average velocity as the curl of its vector potential, $U = \nabla \times A$ and using (4.6), we obtain in the axial gauge:

$$S = a \int d^2x\, dz_1\, dz_2\, e^{-\alpha|z_1 - z_2|}\, [-\epsilon^{AS}\partial_3 A_S\, \epsilon^{RM}\partial_A \partial_R A_M + \tfrac{1}{2}\partial_3(\epsilon^{RS}\partial_R A_S)^2]_{z_1}\, [\ldots]_{z_2}, \tag{4.9}$$

where the second factor in square brackets has the same structure as the first one, but it is evaluated at z_2. We now notice that the second term in both square brackets in the last equation is a total derivative. Consequently, upon integration by parts, its contribution becomes proportional to α and hence it is small for small values of that parameter. Therefore, that term can be omitted without substantially affecting the results and we shall do so in this paper.

We next compute the functional derivatives of (4.9) in order to generate the second term of S_2^1 in eq. (3.2.11). This is a straight forward procedure and we merely quote the result.

$$\frac{\delta^2 S}{\delta A_A(x_1,z_1)\ \delta A_B(x_2,z_2)} = 2a\, e^{-\alpha|z_1 - z_2|} \{2\alpha\, \epsilon(z_1 - z_2)\, \epsilon^{NA} \epsilon^{RB} \partial_R \partial_N \delta^2(x_1 - x_2)$$

$$+ \alpha^2 \epsilon^{NA} \epsilon^{RM} \partial_N \partial_R A_M\, \epsilon^{PB} \epsilon^{QS} \partial_P \partial_Q A_S\, \delta^2(x_1 - x_2)\}. \qquad (4.10).$$

In this equation and from now on, x_1, etc. stand for the transverse components of the position vector, whereas z_1, etc. denote the longitudinal component of the same vector. The symbol $\epsilon(z)$ stands for the sign function, $\epsilon(z) = z/|z|$. Unlike in the expression of S, we now cannot omit terms of $O(\alpha)$ or $O(\alpha^2)$, for there are no terms of O(1) present in eq. (4.10). However, we notice that the quantity α can be scaled out of the expression S_2^1. In fact, on looking at the structure of eqs. (3.1.4), (3.2.11), (4.9) and (4.10),we realize that upon the rescaling, $z \to (1/\alpha)z$, $A \to (1/\alpha^{1/2})A$, and $Z_{AB} \to (1/\alpha^3)Z_{AB}$, the expression of S_2^1 is multiplied by an overall factor $1/\alpha^4$. This, however, can be absorbed into the constant entering the expression of the force correlation function; at any rate, the location of the stationary point of the functional S_2^1 is independent of the value of α. (We remark, however, that higher order approximations to S_2 do not have this scaling property.)

The reader certainly notices that our approach has been quite general and the results obtained so far are applicable to almost any flow geometry. We now specialize to the case of a cylindrically symmetric free jet. Such jets have a characteristic velocity, namely the mean flow velocity at the centerline of the jet exit, U_{0m}, and a characteristic length, d, namely the diameter of the jet at the exit. From now on, as is customary, we measure all velocities and distances in these units. However, in order to keep the notation simple, this is not indicated explicitly in the formulae. (For instance, a velocity component denoted by U is understood to mean U/U_{0m}, or an axial distance, z, is to be interpreted as z/d.)

A cylindrically symmetric mean flow can be described in the axial gauge by means of a vector potential which has a tangential component only. Specifically, in cylindrical coordinates ($x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$), we choose a trial function for the vector potential of

the form:

$$A_\varphi = \tfrac{1}{2}U(0,z)\, R(z)^2\, [1 - \exp(-\, r^2/R(z)^2)], \tag{4.11}$$

with the other two components vanishing. Here U(0,z) stands for the mean velocity along the jet axis; the quantity R(z) characterizes the radius of the jet at distance z from the exit. The choice of such a functional form is motivated by its simplicity and by various fits and approximate calculations of free jets; see, e.g. the classic text of Hinze (1975). This vector potential gives rise to an axial and a radial component of the mean velocity of the form:

$$U_z(r,z) = U(0,z)\, e^{-r^2/R(z)^2},$$

$$\begin{aligned} U_r(r,z) = \{&\frac{d\ln U(0,z)}{dz}\, \tfrac{1}{2}R(z)^2\, (1 - e^{-r^2/R(z)^2}) \\ &+ R(z)\frac{dR}{dz}\, [1 - (1 + \frac{r^2}{R^2})e^{-r^2/R(z)^2}]\}. \end{aligned} \tag{4.12}$$

Next we impose the requirement of cylindrical symmetry on the two–point correlation function, Z_{AB}, cf. eq. (3.1.4):

$$Z_{AB} = \delta_{AB}\, Z_1 + \epsilon^{AB}\, Z_2.$$

(4.13)

In order to simplify the computation, we arbitrarily set $Z_2 = 0$. There is no strong physical motivation for this choice; it simplifies the calculations to a considerable extent, albeit at the cost of some loss of accuracy. We assume the following functional form for Z_1:

$$\begin{aligned} Z_1 = A\, &e^{-f(z_1 - z_2)^2 - g(x_1^A - x_2^A)(x_1^A - x_2^A)} \\ &\times [U(0,z_1)U(0,z_2)]^d\, (1 + BM)\, e^{-\delta M}. \end{aligned} \tag{4.14}$$

In this equation, the parameters A, B, d, δ, f, and g are treated as variational parameters,

although later we found that the results are rather insensitive to the choice of d and we ended up using a plausible value, $d = 1/2$. The expression M is defined as follows:

$$M = \tfrac{1}{2}\left(\frac{\hat{x}_1 \hat{x}_1}{R(z_1)^2} + \frac{\hat{x}_2 \hat{x}_2}{R(z_2)^2}\right). \tag{4.15}$$

The radius of the jet was assumed to be a linear function of z, $R(z) = a + bz$, with the parameters a and b being also varied. We did not, however, vary the functional form of the mean velocity on the jet axis: although this is possible in principle, we used a fixed form in order to simplify the calculational task. We used the following simple modification of Spalding's formula (Hinze 1975), which is quite accurate for $z \geq 2$:

$$U(0,z) = \frac{1.35}{1 + 0.038z^{3/2}}. \tag{4.16}$$

The next task is to insert the expressions of the vector potential of the average flow and of the two point correlation function into the functional $S_2{}^1$ and to minimize that expression with respect to the parameters. The operations are elementary, but extremely tedious to carry out by hand, although this is not impossible. They are, however, easily carried out by a symbolic manipulation program. We carried out the differentiations and a part of the integrations by using the program "Maple", (Char et al. 1985). The search for an extremum of the functional was done numerically. All computations were done on a Sun 3/160 computer. In order to carry out the search for the extremum, one needs starting values of the parameters. These were obtained by comparing $U_z(r,z)$, eq. (4.12), to a few data points from Zoltani and Bicen (1990a); we also chose, arbitrarily, $f = g = \delta = 1$ as starting values. (The values of f and g cannot be read off directly from the data.)

The search resulted in the following values of the parameters:

$$\begin{gathered} a = 0.25, \;\; b = 0.076; \\ A = 0.0013, \; B = 2.3 \\ \delta = 1.79, \\ f \approx 5.6, \; g \approx 3.7 \end{gathered} \tag{4.17}$$

In the approximation used, the value of the functional S_2 changes rather slowly vith variations of the parameters f and g. This is due to the fact that in this approximation, the interaction between the fluctuations and the mean flow is taken into account, but the mutual interaction of the fluctuations is neglected. (The functional $S_2{}^1$ is almost local: it contains δ functions and derivatives of δ functions of finite order.) Hence, the values of f and g cannot be determined efficiently. This is not a serious deficiency, however: most measurements determine correlation functions at coincident arguments, which are rather insensitive to these parameters.

The correlation functions of velocity fluctuations are determined by inserting the expression of Z into eq. (3.1.5). We summarize the results by listing the expressions of the axial component of the mean flow and of the correlation functions at coincident arguments.

$$
\begin{aligned}
G_{11} = G_{22} = 2fA\, U_z(0,z)\, e^{-\chi^2} \{1 &+ B\chi^2 + (1/2f)[d(z)^2 \\
&+ \chi^2\, B\, d(z)\, (d(z) - (B-1)/B\; b/R(z)) \\
&+ \chi^4\, B\, b/R\, (d(z) + (1-2B)/4B\; b/R(z)) \\
&+ \chi^5 B/4\, (b/R(z))^2]\}, \\
G_{33} = 4gA\, U_z(0,z)\, e^{-\chi^2} \{1 &+ \chi^2[B - (2B-1)/4R(z)^2 g] \\
&+ \chi^4 B/4R(z)^2 g\}, \\
G_{13} = G_{23} = A\, \chi\, e^{-\chi^2}\, U(0,z)\, \{ &(1-B)\, d(z) \\
&+ \chi^2\, [B\, d(z) + (1-2B)\, b/R] + \chi^4\, bB/R(z)\}, \\
U_z(r,z) = U(0,z)\, e^{-\chi^2}&.
\end{aligned}
\tag{4.18}
$$

Here, $\chi^2 = \delta r^2/R(z)^2$ and $d(z) = d/dz(U(0,z))^{1/2}$.

The results listed in eq. (4.18) are evaluated using the parameter values (4.17) and compared with data taken from Zoltani and Bicen (1990a), (1990b) in Figures 1 thru 4. (The experimental circumstances there were not exactly identical. In particular, the Reynolds numbers differ by approximately 30% at the jet exit. However, at large Reynolds numbers the profile of the mean flow and the correlation functions should be insensitive to the precise value of Re, cf. the discussion at the beginning of this Section. The data bear out this conclusion.) In the figures we use conventional notation. The correspondence

between the notation used in this paper (better suited for theoretical calculations) and the conventional one, as used in Zoltani and Bicen (1990a), (1990b) is the following. (All arguments are suppressed, so, for instance, G_{11} stands for $G_{11}(\mathbf{x},\mathbf{x})$. This does not lead to any ambiguity.)

$$\begin{aligned} G_{33} &= \frac{<u^2>}{U_{0m}{}^2}, \\ G_{11} &= \frac{<v^2>}{U_{0m}{}^2}, \\ G_{22} &= \frac{<w^2>}{U_{0m}{}^2}, \\ G_{13} &= \frac{<uv>}{U_{0m}{}^2}, \\ U_z &= U \end{aligned} \tag{4.19}$$

5. DISCUSSION

A look at Figures 1 thru 4 shows that overall, the simple trial functions used in our variational calculation, with the parameters determined by extremizing S_2, are in a reasonable agreement with the data. (The agreement with the data is better at large values of z. This is understandable on physical grounds: at larger values of z, the statistics of the flow is closer to the stationary ensemble we have been working with.) In particular, our assumption of cylindrically symmetric correlation functions appears to be well justified. This is expected on physical grounds: to a very good approximation, the mean flow is cylindrically symmetric. Although this does not necessary imply that the correlation functions themselves should possess the symmetry, one expects violations of this symmetry to occur on relatively short time scales. Likewise, the assumption about "transverse scaling", namely that the radial dependence of both the mean flow and the correlation functions occurs only in the scale invariant combination, $\chi^2 = \delta r^2/R(z)^2$ appears to be reasonably well satisfied.

A notable exception is the correlation function G_{13}, which is predicted to be much too small compared to the data. Various modifications of the form of the trial functions have

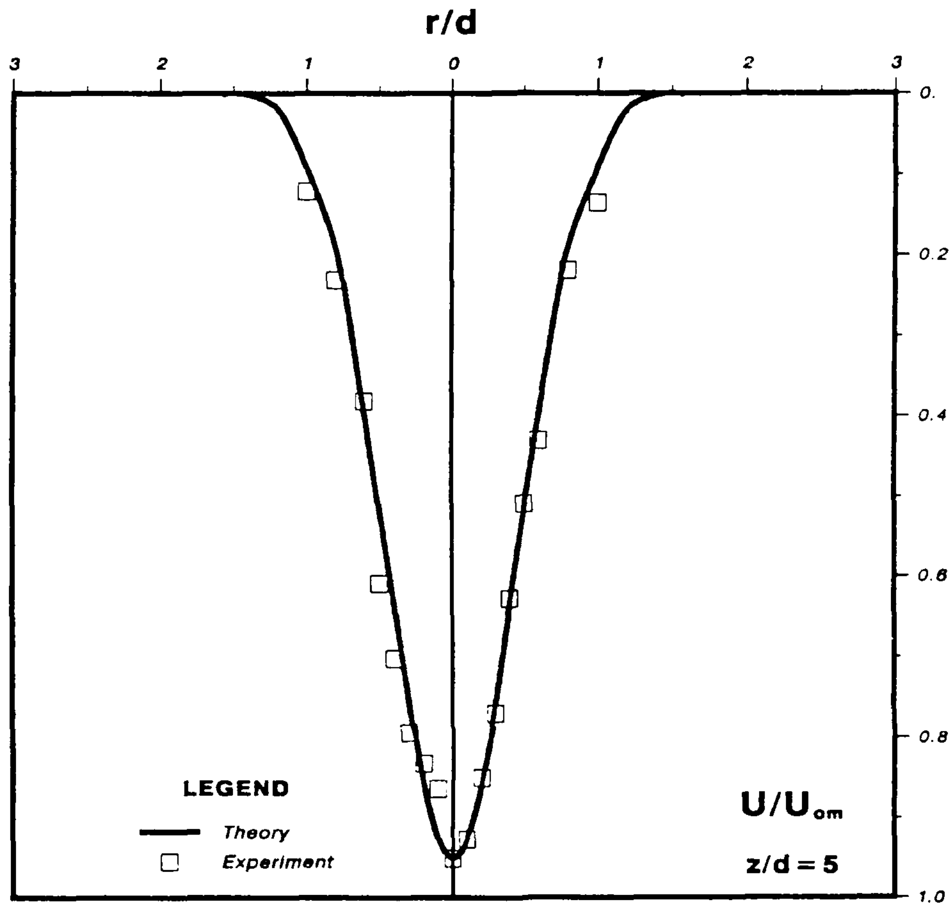

Figure 1. Predicted and Measured Mean Velocity Profiles at (z/d) = 5. The Values Were Normalized With the Centerline Jet Exit Velocity.

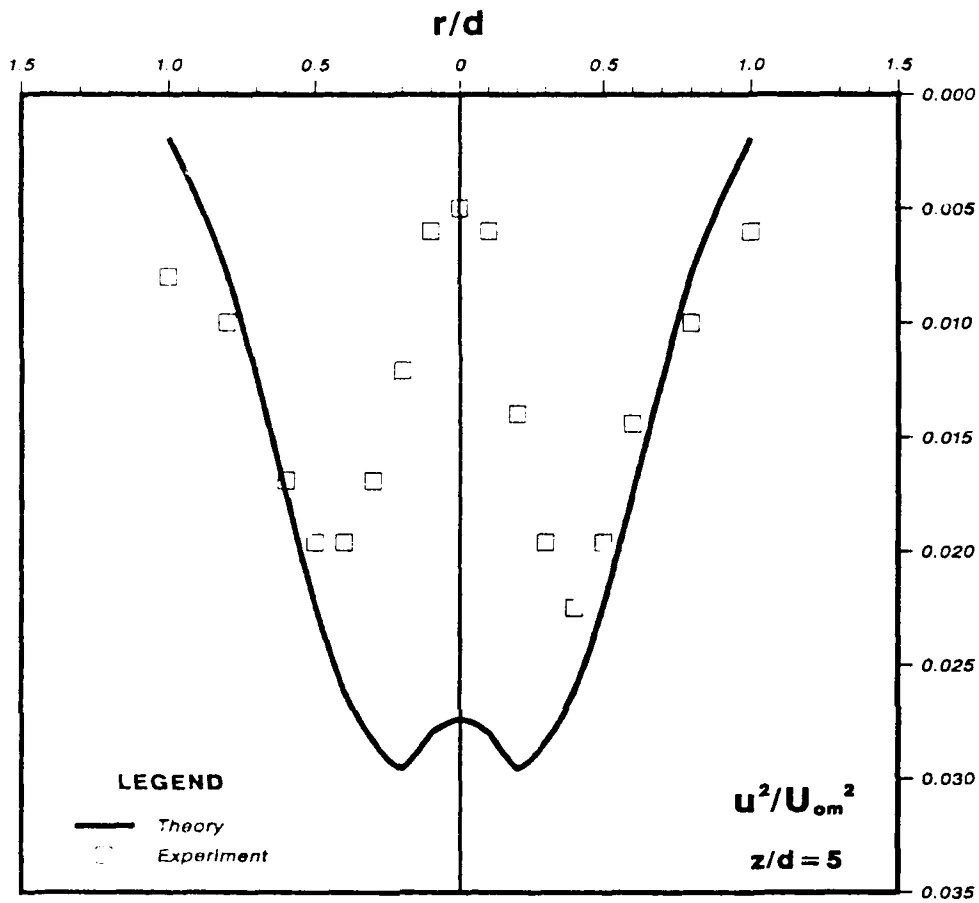

Figure 2. Predicted and Measured Values of the Autocorrelation Function, $<u^2>$ in the Axial Direction at (z/d) = 5. The Values Were Normalized With the Square of the Centerline Jet Exit Velocity.

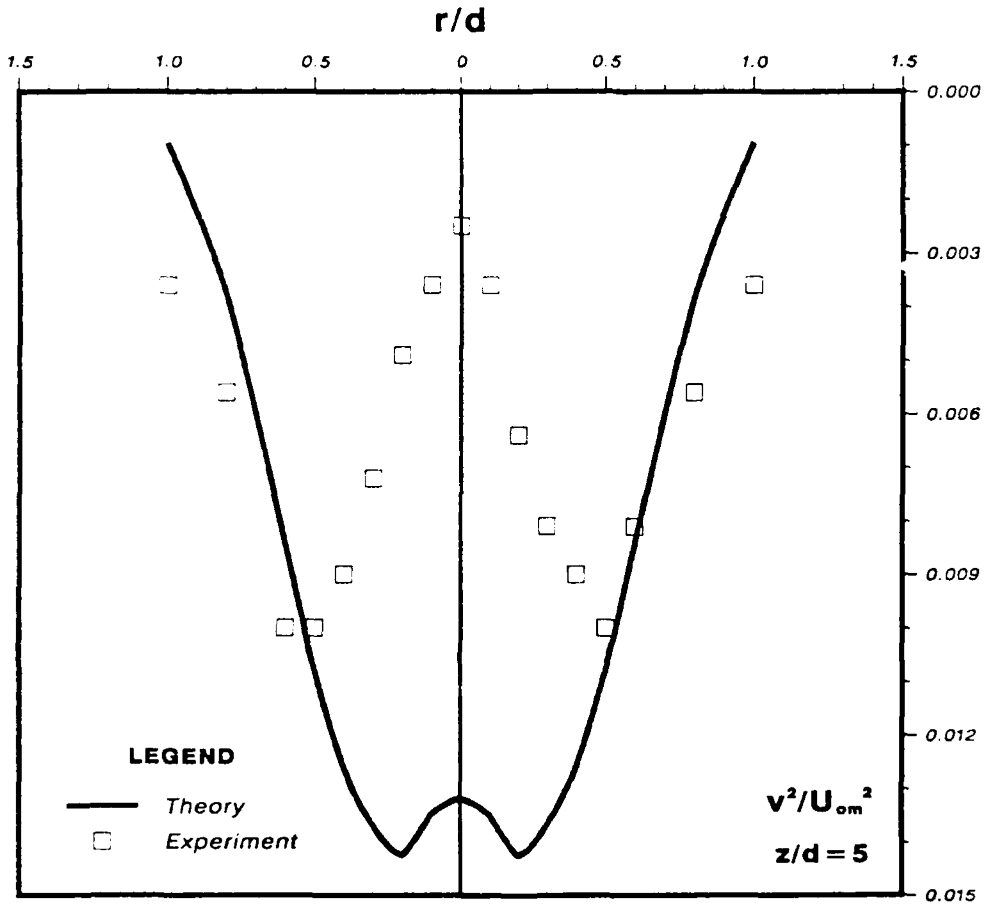

Figure 3. Predicted and Measured Autocorrelation Function, $<v^2>$ in the Radial Direction at (z/d) = 5. The Values Were Normalized With the Square of the Centerline Jet Exit Velocity.

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